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REDUCING PRIME GRAPHS AND RECOGNIZING CIRCLE GRAPHS

ANDRÉ BOUCHET

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We prove a reduction theorem for prime (simple) graphs in Cunningham's sense. Roughly speaking this theorem says that every prime (simple) graph of order $n > 5$ "contains" a smaller prime graph of order $n-1$. As an application we give a polynomial algorithm for recognizing circle graphs.

1. Introduction

Let us consider a word m such that each letter occuring in m occurs precisely twice in m. We say that m is a *double occurence word.* An *alternance* of m is a pair vw of distinct letters such that we meet alternatively $...v...w...v...w...$ when reading m. The *alternance graph* $A(m)$ is the simple graph whose vertices are the letters of m and whose edges are the alternances of m. Alternance graphs were used in [2] for an algorithmic solution of the Gauss problem on self-intersecting curves of the plane. The same family of graphs is related to stack sorting techniques in a paper by Even and Itai [6]. From a geometric point of view, alternance graphs can be interpreted as intersection graphs of chords of a circle, and they are often called *circle graphs.* The reader will find in [9] a survey of some results known on circle graphs.

Two basic questions are still unsolved for circle graphs. Is there a good characterization of circle graphs? Does there exist a polynomial algorithm for recognizing circle graphs? (Note that a polynomial algorithm is in particular a good characterization.) We discuss the characterization problem for introducing the subject of this paper.

Characterizations of circle graphs have already been found by Fournier [7] and de Fraysseix [8], but they do not yield a polynomial algorithm for proving that a given graph is not a circle graph, and thus are not good characterizations in Edmonds' sense. Our work on isotropic systems [3, 4] shows that circle graphs are precisely the fundamental graphs of the graphic systems, where the graphic systems are the analog of the graphic matroids in the theory of matroids, and the fundamental graphs are the analog of the fundamental Whitney's representations of matroids. Going on with this analogy suggests the possibility of finding a good characterization analog to

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Tutte's theorem [12] on excluded minors of graphic matroids. The analog of minors is the following.

If G is a (simple) graph and v is a vertex of G, let us set $n(v)=\{w: vw\in E(G)\}$. A *local complementation* of G at v consists in replacing the subgraph induced by G on $n(v)$ by the complementary subgraph. Two graphs are *locally equivalent* if one of them can be obtained from the other by a succession of local complementations. An *i-minor* of G is a subgraph of a graph locally equivalent to G.

It is well known that circle graphs are stable by local complementation. Indeed if G is the alternance graph of a word vm_1vm_2, where v is a letter and m_1 and m_2 are subwords, then a local complementation of G at v gives the alternance graph of $vm_1v\tilde{m}_2$, where $\tilde{m}_2$ is obtained by reversing m_2. It is obvious that a subgraph of a circle graph is a circle graph. Therefore every i-minor of a circle graph is a circle graph. Thus the analog of Tutte's theorem is the following.

Conjecture 1.1. *There exists a finite set M of graphs which are not circle graphs such that every graph which is not a circle graph has an i-minor isomorphic to a graph in M.*

This conjecture would clearly yield a good characterization of circle graphs.

The purpose of this paper is to give a solution to the second question with a polynomial algorithm for recognizing circle graphs and, if possible, realizing it by a double occurence word. It uses the theory of graph decompositions in Cunningham's sense [5]. A *split* of a (simple) graph G is a bipartition $\{V_1, V_2\}$ of the vertex-set satisfying

(i) $|V_1|, |V_2|\geqq 2$,

(ii) there exist $W_1\subseteq V_1$ and $W_2\subseteq V_2$ such that $\{v_1v_2\in E(G): v_1\in V_1, v_2\in V_2\}= =\{v_1v_2: v_1\in W_1, v_2\in W_2\}$. A *prime* graph is a graph without split. Our algorithm is essentially based on a "reduction theorem of prime graphs" which is the main result of this paper.

Theorem. *Every prime graph of order $n>5$ has an i-minor of order $n-1$ which is also prime.*

The paper is presented in order to be independent of the theory of isotropic systems, except Property (5.2) for which we have at present no purely graph theoretic proof. In a subsequent paper about connectivity of isotropic systems, it will be shown that the reduction theorem for prime graphs is similar to Tutte's "wheel and whirl theorem" for reducing 3-connected matroids [11]. Thus our algorithm for recognizing circle graphs will appear to be similar to Inukai and Weinberg's algorithm [10] for recognizing graphic matroids.

2. Decomposition of graphs

If p is a word on a set V we shall denote by $\tilde{p}$ the word obtained by reversing the sequence of the letters. The concatenation of two words p and q is denoted by pq. The empty word is denoted by Λ.

The graphs considered are simple. Undefined notions and notations follow Bondy and Murty [1]. The decompositions of graphs which we study are those introduced by Cunningham [5], and our notation and definitions follow in general his paper at the exception of the composition sign which will be c instead of $*$ which is used for local complementation.

If G is a graph and v is a vertex of G, then $G\setminus v$ is the subgraph obtained by deleting v. If X and Y are disjoint subsets of vertices, then we let $[X, Y]_G=\{vw\in E(G): v\in X,\ w\in Y\}$. The edge-set of the complete bipartite graph on the color-classes X and Y is denoted by $K(X, Y)$. A *split* of G is a bipartition $\{V_1, V_2\}$ of $V(G)$ satisfying:

(i) $|V_1|, |V_2|\geqq 2$,

(ii) there exist subsets $W_1\subseteqq V_1$ and $W_2\subseteqq V_2$ such that $[V_1, V_2]_G=K(W_1, W_2)$.

The *neighborhood* of v in G is $n(v)=\{w: vw\in E(G)\}$. The *local complementation* of G at v consists in replacing the subgraph induced by G on $n(v)$ by the complementary subgraph. We shall denote by $G*v$ the graph which is so obtained. Obviously $(G*v)*v=G$. If $v_1v_2...v_q$, $q>1$, is a word on V we define recursively $G*v_1v_2...v_q=(G*v_1v_2...v_{q-1})*v_q$. We let $G*\Lambda=G$. Two graphs G and H are said to be *locally equivalent* if $V(G)=V(H)$ and there exists a word p on $V(G)$ such that $H=G*p$. This is actually an equivalence relation because $G*\Lambda=G$, $(G*p)*q=G*pq$, and $H=G*p\Rightarrow G=H*\tilde{p}$ are satisfied whenever p and q are words on $V(G)$.

Lemma 2.1. *Locally equivalent graphs have the same splits.*

Proof. Let G be a graph, $v\in V(G)$, and let $\{V_1, V_2\}$ be a split of G. The property will be proved after verifying that $\{V_1, V_2\}$ is a split of $G'=G*v$. We suppose that $v\in V_1$, and we let $[V_1, V_2]_G=K(W_1, W_2)$. If $v\notin W_1$ then locally complementing G at v does not change any edge of $[V_1, V_2]_G$; therefore $\{V_1, V_2\}$ remains a split of G'. If $v\in W_1$ then $W_2\subseteqq n(v)$. When locally complementing G at v, each edge of $K(n(v)\cap W_1, W_2)$ disappears when each edge of $K(n(v)\cap(V_1\setminus W_1), W_2)$ is created. Therefore we have $[V_1, V_2]_{G'}=K(W_1\triangle(n(v)\cap V_1), W_2)$, and $\{V_1, V_2\}$ is still a split of G'. ∎

Two graphs G_1 and G_2 are *composable* if $|V(G_1)|, |V(G_2)|\geqq 3$ and $|V(G_1)\cap V(G_2)|=1$. The common vertex v of G_1 and G_2 is called the *marker*. The *composition* of G_1 and G_2 is the graph $G=G_1cG_2$ defined by $V(G)=V(G_1\setminus v)\cup V(G_2\setminus v)$, $E(G)=E(G_1\setminus v)\cup E(G_2\setminus v)\cup\{v_1v_2: v_1v\in E(G_1),\ vv_2\in E(G_2)\}$. If we let $V_1=V(G_1\setminus v)$ and $V_2=V(G_2\setminus v)$, then $\{V_1, V_2\}$ is a split of G. Conversely, if $\{V_1, V_2\}$ is a split of a graph G, and if v is an element not in $V(G)$ then there exists precisely one pair $\{G_1, G_2\}$ of composable graphs with marker v such that $G_1cG_2=G$. A graph is said to be *prime* if it has no split and at least three vertices. We notice that a prime graph is necessarily connected. Property (2.1) obviously implies.

Lemma 2.2. *A graph locally equivalent to a prime graph is prime.* ∎

A finite set D of graphs is *composable* if the graph constructed on D as a vertex-set and whose edges are the composable pairs extracted from D is a tree. A *partial composition* of D consists in replacing a composable pair extracted from D by its composition, obtaining so a new composable set of graphs. We denote by D^c the graph obtained after performing $|D|-1$ partial compositions (D^c does not depend of the order of the compositions). A *canonical decomposition* of a graph is a composable set D of graphs satisfying:

(i) every graph in D is prime or complete,

(ii) no two complete graphs of D are composable,

(iii) $D^c=G$.

Theorem 2.3. (Cunningham [5]) *The canonical decomposition of a graph is unique up to a bijection between the markers of the composable pairs. If G is of order n, the canonical decomposition can be obtained by a polynomial algorithm recquiring $O(n^3)$ time and $O(n^2)$ space (and so this algorithm recognizes in particular whether G is prime).* ▮

3. Reduction theorem for prime graphs

An *i-minor* of a graph G is a subgraph of a graph locally equivalent to G. An i-minor H is said to be *elementary* if it is obtained by deleting one vertex to a graph locally equivalent to G. So we can write $H=G*p\setminus v$ where p is a word on $V(G)$ and $v\in V(G)$.

Lemma 3.1. *A prime graph of order 5 is locally equivalent to a 5-cycle.*

Proof. Such a graph G has no vertex of degree 0 because it is connected. It has no vertex x of degree 1 because if y is the neighbour of x, $\{x, y\}$ is a split of G. Let us suppose first that G has no vertex of degree 2, so that all the vertices are of degree 3 or 4. There is at least one vertex x of degree 4 since otherwise the sum of the degrees would be odd. The subgraph induced on $V-x$ has all its vertices of degree 2 or 3, and so it is a 4-cycle C with 0, 1 or 2 diagonals. In each case, if a, b, c, d are the successive vertices of C, $\{\{a, c\}, \{x, b, d\}\}$ is a split of G. Therefore there is a vertex y of degree 2. We can suppose that a and b, the neighbours of y, are not joined by an edge, since otherwise we could locally complement at y for suppressing this edge. Therefore a and b have degrees <4, and this is also true for the vertices c and d which are not joined to y. So we have 0, 2 or 4 vertices of degree 3. If there are 4 vertices of degree 3, then $\{\{y, a, b\}, \{c, d\}\}$ is a split of G. If there are 2 vertices of degree 3, G is a 5-cycle with one diagonal which can be suppressed by locally complementing G at the vertex which is joined to the ends of this diagonal. If no vertex is of degree 3, G is a 5-cycle. ▮

Theorem 3.2. *Every prime graph of order $n>5$ has some elementary i-minor which is prime.*

Proof. Let G be such a graph on the vertex-set V. We suppose that the theorem is false and we seek for a contradiction.

We choose a vertex v and we let $H=G*v$. Since $G\setminus v$ and $H\setminus v$ are not prime we can determine disjoint subsets X_1, X_2, X_3, X_4 of $V-v$ such that $\{X_1\cup X_3, X_2\cup X_4\}$ is a split of $G\setminus v$ and $\{X_1\cup X_2, X_3\cup X_4\}$ is a split of $H\setminus v$. For $i=1, 2, 3, 4$ we define the subsets a_i, b_i and c_i of X_i such that $[X_1\cup X_3, X_2\cup X_4]_G=K(a_1\cup a_3, a_2\cup a_4)$, $[X_1\cup X_2, X_3\cup X_4]_H=K(b_1\cup b_2, b_3\cup b_4)$ and c_i is the set of the vertices of G (or equivalently of H) which are joined to v in X_i. The pairs $\{X_1, X_4\}$ and $\{X_2, X_3\}$ are called the diagonals w.r.t. Fig. 1 where the split of G is separated by the vertical line when the split of H is separated by the horizontal line.

1. *There exists a diagonal $\{X_i, X_j\}$ such that $c_i\neq\emptyset$ and $c_j\neq\emptyset$.*

Proof. On the contrary one of the subsets $c_1\cup c_2$, $c_2\cup c_4$, $c_4\cup c_3$, $c_3\cup c_1$ is empty. If for example $c_1\cup c_2=\emptyset$ then $[X_1\cup X_2,\ X_3\cup X_4\cup\{v\}]_H=[X_1\cup X_2,\ X_3\cup X_4]_H\cup$

$$\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}$$

Fig. 1

$\cup[X_1\cup X_2,\ \{v\}]_H=[X_1\cup X_2,\ X_3\cup X_4]_H=K(b_1\cup b_2,\ b_3\cup b_4)$, a contradiction because H has no split. The other cases are treated similarly. ∎

2. $a_1\cup a_3\neq c_1\cup c_3,\ a_2\cup a_4\neq c_2\cup c_4,\ b_1\cup b_2\neq c_1\cup c_2,\ b_3\cup b_4\neq c_3\cup c_4.$

Proof. Let us suppose that $a_1\cup a_3=c_1\cup c_3$. Then we can write

$$\begin{aligned}[X_1\cup X_3, X_2\cup X_4\cup\{v\}]_G &= \\ &= [X_1\cup X_3, X_2\cup X_4]_G\cup[X_1\cup X_3, \{v\}]_G \\ &= K(a_1\cup a_3, a_2\cup a_4)\cup K(c_1\cup c_3, \{v\}) \\ &= K(a_1\cup a_3, a_2\cup a_4\cup\{v\}),\end{aligned}$$

a contradiction because G has no split. The other cases are dealt similarly. ∎

3. *If $\{X_i, X_j\}$ is a diagonal then one of the six following cases must occur: $a_i=\emptyset$ or $a_j=\emptyset$ or $c_i=\emptyset$ or $c_j=\emptyset$ or $a_i=c_i$ or $a_j=c_j$. Moreover the following six implications hold:*

$$a_i=\emptyset \Rightarrow b_i=c_i \text{ and } b_j=c_j, \tag{3.1}$$

$$a_j=\emptyset \Rightarrow b_i=c_i \text{ and } b_j=c_j, \tag{3.2}$$

$$c_i=\emptyset \Rightarrow a_i=b_i \text{ and } a_j=b_j, \tag{3.3}$$

$$c_j=\emptyset \Rightarrow a_i=b_i \text{ and } a_j=b_j, \tag{3.4}$$

$$a_i=c_i \Rightarrow a_i=b_i=c_i \text{ and } a_j\triangle b_j\triangle c_j=\emptyset \text{ except if } b_i=b_j=\emptyset, \tag{3.5}$$

$$a_j=c_j \Rightarrow a_j=b_j=c_j \text{ and } a_i\triangle b_i\triangle c_i=\emptyset \text{ except if } b_i=b_j=\emptyset. \tag{3.6}$$

Proof. We have $[X_i, X_j]_G=K(a_i, a_j)$, and the set of edges $[X_i, X_j]_H$ is obtained from $[X_i, X_j]_G$ after a local complementation at v, so that $[X_i, X_j]_H=K(a_i, a_j)\triangle\triangle K(c_i, c_j)$. We have

$$K(a_i, a_j)=K(a_i\setminus c_i, a_j\setminus c_j)\triangle K(a_i\setminus c_i, a_j\cap c_j)\triangle K(a_i\cap c_i, a_j\setminus c_j)\triangle K(a_i\cap c_i, a_j\cap c_j)$$

and

$$K(c_i, c_j)=K(c_i\setminus a_i, c_j\setminus a_j)\triangle K(c_i\setminus a_i, c_j\cap a_j)\triangle K(c_i\cap a_i, c_j\setminus a_j)K(c_i\cap a_i, c_j\cap a_j).$$

Therefore $[X_i, X_j]_H$ is equal to the union of the six pairwise disjoint sets of edges represented by the sides of the hexagon of Fig. 2, with the convention that a side between vertices labelled by sets E and F represents $K(E, F)$. The sets which label the vertices of the hexagon are pairwise disjoint. Therefore $[X_i, X_j]_H$ will be the edge-set of a complete bipartite graph if and only if there exists two empty sets corresponding to two vertices at distance 2 in the hexagon. The six possibilities for choosing

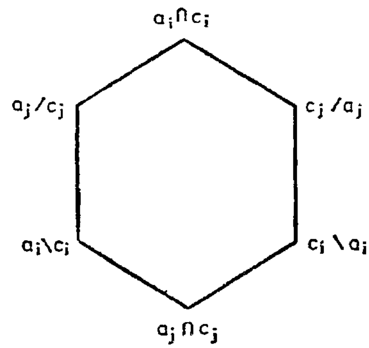

Fig. 2

these two sets yield the six cases of the statement, and it is then easy, to verify each of the six implications. ∎

4. *No subset X_i is such that $a_i = b_i = c_i = \emptyset$.*

Proof. Suppose for example that $a_1 = b_1 = c_1 = \emptyset$. We have $a_3 \neq \emptyset$ since otherwise $[X_1 \cup X_3, X_2 \cup X_4 \cup \{v\}]_G = [X_1 \cup X_3, \{v\}]_G = K(c_1 \cup c_3, \{v\})$, and G has a split. Similarly $b_2 \neq \emptyset$. Since $c_1 = \emptyset$, Property (1) implies $c_2 \neq \emptyset$ and $c_3 \neq \emptyset$. Since $a_1 = c_1 = \emptyset$, Property (2) implies $a_3 \neq c_3$. Similarly $b_1 = c_1 = \emptyset$ implies $b_2 \neq c_2$. Therefore we have $\emptyset \neq a_3 \neq c_3 \neq \emptyset$ and $\emptyset \neq b_2 \neq c_2 \neq \emptyset$ which are incompatible with the six conditions of Property (3). ∎

5. *A subset X_i is constituted of a single vertex if either $a_i = \emptyset$ or $c_i = \emptyset$ or $a_i = c_i$.*

Proof. We do not lose in generality in supposing $i = 1$.

Let us suppose that $a_1 = \emptyset$. Property (3.1) implies $b_1 = c_1$. Therefore $[X_1, X_3]_H = K(b_1, b_3) = K(c_1, b_3)$. After locally complementing at v, the set of edges $K(c_1, b_3) = [X_1, X_3]_H$ becomes $K(c_1, b_3 \triangle c_3) = [X_1, X_3]_G$. Since $a_1 = \emptyset$ we can write

$$\begin{aligned}
[X_1, X_2 \cup X_3 \cup X_4 \cup \{v\}]_G = \\
&= [X_1, X_2 \cup X_3 \cup X_4]_G \cup [X_1, \{v\}]_G \\
&= [X_1, X_3]_G \cup [X_1, \{v\}]_G \\
&= K(c_1, b_3 \triangle c_3) \cup K(c_1, \{v\}) \\
&= K(c_1, b_3 \triangle c_3 \triangle \{v\}).
\end{aligned}$$

The pair $\{X_1, X_2 \cup X_3 \cup X_4 \cup \{v\}\}$ cannot be a split of G because G is prime. Therefore we have either $|X_1| \leq 1$ or $|X_2 \cup X_3 \cup X_4 \cup \{v\}| \leq 1$. Following (4) no X_i is empty. Therefore we must have $|X_1| = 1$.

Suppose now that $c_1 = \emptyset$. Property (3.3) implies $a_1 = b_1$, and so $[X_1, X_3]_H = K(b_1, b_3) = K(a_1, b_3)$. After locally complementing H at v, the set of edges $[X_1, X_3]_H$ remains unchanged because $c_1 = \emptyset$, and so $[X_1, X_3]_G = K(a_1, b_3)$. The set of edges $[X_1, \{v\}]_G$ is empty because $c_1=\emptyset$. Therefore we can write

$$\begin{aligned}
[X_1, X_2\cup X_3\cup X_4\cup\{v\}]_G &= \\
&= [X_1, X_3]_G\cup[X_1, X_2\cup X_4]_G\cup[X_1, \{v\}]_G \\
&= K(a_1, b_3)\cup K(a_1, a_2\cup a_4) \\
&= K(a_1, a_2\cup b_3\cup a_4).
\end{aligned}$$

Since $\{X_1, X_2\cup X_3\cup X_4\cup\{v\}\}$ cannot be a split of G, we conclude as above $|X_1|=1$.

Let us suppose finally that $a_1=c_1$. If b_1 and b_4 are nonempty then property (3.5) implies $a_1=b_1=c_1$, and so $[X_1, X_3]_H=K(b_1, b_3)=K(a_1, b_3)$. Therefore after locally complementing H at v, we obtain $[X_1, X_3]_G=K(a_1, b_3\triangle c_3)$. So we obtain similarly as above

$$[X_1, X_2\cup X_3\cup X_4\cup \{v\}]_G = K(a_1, a_2\triangle b_3\triangle c_3\triangle a_4\triangle \{v\}),$$

and we conclude to $|X_1|=1$. If $b_1=b_4=\emptyset$ then $[X_1, X_3]_H=\emptyset$, and after locally complementing at v we obtain $[X_1, X_3]_G=K(c_1, c_3)=K(a_1, c_3)$, so that

$$[X_1, X_2\cup X_3\cup X_4\cup\{v\}]_G = K(a_1, a_2\triangle c_3\triangle a_4\triangle \{v\}),$$

and again we conclude to $|X_1|=1$. ∎

6. *There is a graph locally equivalent to G which has a vertex of degree 2.*

Proof. It follows from (3) and (5) that each diagonal contains a subset X_i which is reduced to a single vertex. Considering the diagonals together we shall find two subsets X_i and X_j reduced to single vertices. The pair of subsets $\{X_i, X_j\}$ is not itself a diagonal, and so we can suppose without loss of generality that $X_i=X_1$ and $X_j=X_3$. We denote by x_1 (x_3) the vertex of X_1 (X_3).

If $a_1=\emptyset$ then $[X_1, X_2\cup X_4]_G=\emptyset$, and so there are at most two edges incident to x_1 in G, the edges x_1x_3 and x_1v. These two edges exist otherwise G would not be prime. Therefore x_1 is of degree 2 in G.

If $a_3=\emptyset$ we prove similarly that x_3 is of degree 2 in G.

We suppose now that $a_1\neq\emptyset$ and $a_3\neq\emptyset$. Therefore $a_1=\{x_1\}$ and $a_3=\{x_3\}$. It is impossible that both $c_1\neq\emptyset$ and $c_3\neq\emptyset$ since otherwise we should have $c_1\cup c_3=a_1\cup a_3$, a contradiction with (2). It is impossible that $c_1=c_3=\emptyset$ by (1). Therefore we have either $c_1\neq\emptyset$ and $c_3=\emptyset$ or $c_1=\emptyset$ and $c_3\neq\emptyset$. We make the first assumption, the second being symmetric.

Suppose that x_1x_3 is an edge of G. Then the neighborhood of x_3 in G is $n(x_3)=\{x_1\}\cup a_2\cup a_4$, and the neighborhood of x_1 in the subgraph induced by G on $n(x_3)$ is $a_2\cup a_4$. Therefore after locally complementing at x_3 we suppress the edges of $K(\{x_1\}, a_2\cup a_4)$, obtaining so a graph locally equivalent to G where x_1 is of degree 2.

Suppose now that x_1x_3 is not an edge of G. Choose a vertex x in $a_2\cup a_4$ (if $a_2\cup a_4$ was empty G would not be connected) and make a local complementation at x. The edge x_1x_3 is created and we find again the preceding case. ∎

7. *There is a graph locally equivalent to G with a cycle C of length 5 containing at most two vertices p and q of degree >2. Moreover p and q are at distance 2 on C.*

Proof. Following (6) we can choose v and G such that v is of degree 2 in G. Therefore there are at most two sets X_i and X_j for which c_i and c_j are nonempty. Following (1) $\{X_i, X_j\}$ is a diagonal. We suppose $c_1 \neq \emptyset$, $c_4 \neq \emptyset$, $c_2 = c_3 = \emptyset$. This implies by (5) that X_2 and X_3 are reduced to single points, say x_2 and x_3. Then (3.3) and (4) imply $a_2 = b_2 = \{x_2\}$ and $a_3 = b_3 = \{x_3\}$. The diagonal $\{X_1, X_4\}$ satisfies one of the six conditions of (3), and thus (5) implies that either X_1 or X_4 is reduced to a single point. These cases are symmetric, and we assume that X_1 is reduced to a single point x_1. In X_4 we denote by x_4 the point which is joined to v, and so $c_4 = \{x_4\}$. Moreover we have $|X_4| > 1$ since $n > 5$. Therefore among the six conditions of (3) for the diagonal $\{X_1, X_4\}$ we can only have $a_1 = \emptyset$ or $a_1 = c_1 = \{x_1\}$.

If $a_1 = \emptyset$ we have $b_1 = c_1 = \{x_1\}$ and $b_4 = c_4 = \{x_4\}$ by (3.1). Equalities $b_1 = \{x_1\}$ and $b_3 = \{x_3\}$ imply $[X_1, X_3]_H = \{x_1 x_3\}$. Equalities $b_2 = \{x_2\}$ and $b_4 = \{x_4\}$ imply $[X_2, X_4]_H = \{x_2 x_4\}$. Since $c_2 = c_3 = \emptyset$, we obtain after locally complementing at v, $[X_1, X_3]_G = \{x_1 x_3\}$ and $[X_2, X_4]_G = \{x_2 x_4\}$. Since $a_2 = \{x_2\}$ and $a_3 = \{x_3\}$ we have $[X_2 X_3]_G = \{x_2 x_3\}$. Since $a_1 = \emptyset$ we have $[X_1, X_2 \cup X_4]_G = \emptyset$. It follows that the cycle $C = (v, x_1, x_3, x_2, x_4)$ and the vertices $p = x_3$ and $q = x_4$ verify the statement.

If $a_1 = c_1 = \{x_1\}$ we have $a_1 = b_1 = c_1 = \{x_1\}$ by (3.5), and we can prove as above that $[X_1, X_3]_G = \{x_1 x_3\}$. Since $a_1 = \{x_1\}$ and $a_3 = \{x_3\}$ the neighborhood of x_3 in G is $n(x_3) = \{x_1, x_2\} \cup a_4$ and the neighborhood of x_1 in $n(x_3)$ is $\{x_2\} \cup a_4$. Therefore after locally complementing at x_3 we suppress the edges of $K(\{x_1\}, \{x_2\} \cup a_4)$, and we obtain so a new graph G' locally equivalent to G satisfying to $a_1' = \emptyset$ as in the preceding case. ∎

We suppose now that G satisfies Property (7), and we assume that v is the vertex joined to p and q. We denote by a and b the two other vertices of C, and we assume that ap and bq are edges. The subset $W = V(G) \setminus \{v, p, q, a, b\}$ is not empty because $n > 5$. Every vertex of W is contained in some path from p to q unless G is not connected or has some 1-vertex cut at p or q. We have already noticed that a prime graph is necessarily connected. Similarly a prime graph G of order $n > 5$ cannot have a 1-vertex cut $\{w\}$ because if Z is a component of $G \setminus w$, then either $\{Z, V \setminus Z\}$ or $\{Y, V \setminus Y\}$, where $Y = Z \cup \{w\}$, is a split. Thus every vertex of W is contained in a path from p to q.

Let $\{X, Y\}$ be a split of $G \setminus v$. Since $c_1 \cup c_2 \cup c_3 \cup c_4 = \{p, q\}$, it follows from (1) that p and q are not in the same member of $\{X, Y\}$. We assume $p \in X$ and $q \in Y$.

Suppose first that a and b are in X. There exists a vertex $w \in W$ which is in Y since otherwise Y would contain only the vertex q, a contradiction because $|Y| \geqq 2$. Since $p \in X$ and $q \in Y$, a path from p to q containing w will contain an edge $w_1 w_2$ such that $w_1 \in X$, $w_2 \in Y$, $w_2 \neq q$. So the two edges $w_1 w_2$ and bq belong to $[X, Y]_G$, and this implies $bw_2 \in [X, Y]_G$ because $[X, Y]_G$ is the edge-set of some complete bipartite graph, a contradiction since no edge can join b to W.

If a and b are in Y, the argument is similar.

If $a \in X$ and $b \in Y$ we have $ab \in [X, Y]_G$. Since $p \in X$ and $q \in Y$ a path joining p to q and disjoint of C contains an edge $w_1 w_2$ satisfying either $w_1 \neq p$ or $w_2 \neq q$. Thus $w_1 b$ and aw_2 are edges of $[X, Y]_G$, a contradiction since no edge joins $\{a, b\}$ to W.

If $a \in Y$ and $b \in X$, the argument is similar.

Finally we obtain always a contradiction. ∎

4. Alternance graphs

A *double occurence word* is a word where each letter appears precisely twice. We do not distinguish double occurence words which are *cyclically equivalent*, i.e. one of them is obtained from the other one by a cyclic permutation of the sequence of letters eventually followed by a reversion. The set of the letters of a double occurrence word will be denoted by $V(m)$. An *alternance* of m is a pair vw of distinct letters such that we meet altenatively $\ldots v \ldots w \ldots v \ldots w \ldots$ when reading m. The *alternance graph* $A(m)$ is defined on the vertex-set $V(m)$, and its edges are the alternances of m. If G is an alternance graph, every double occurence word m such that $A(m)=G$ will be said to *realize* G. In general m is not unique. For example the words abcdeabcde and acebdacebd are not cyclically equivalent, but they have the same alternances.

If m is a word, we recall that $\tilde{m}$ is the word obtained after reversing m.

Two double occurence words m_1 and m_2 are *composable* if $|V(m_1)\cap V(m_2)|=1$. The common letter v of m_1 and m_2 is called the *marker*. A *composition* of m_1 and m_2 is a double occurence word obtained by replacing the two occurences of v in m_1 by the two subwords delimited in m_2 by the occurences of v. Thus if we write $m_1=vA_1vB_1$, $m_2=vA_2vB_2$, where A_1, B_1, A_2 and B_2 are subwords, we can obtain $A_2A_1B_2B_1$, $B_2A_1A_2B_1$, $\tilde{A}_2A_1\tilde{B}_2B_1$, $\tilde{B}_2A_1\tilde{A}_2B_1$. These words are in general not cyclically equivalent, and thus we cannot define an unique composition m_1cm_2. However it is clear that composing m_2 with m_1 gives the same results as composing m_1 with m_2.

(4.1) *The words m_1 and m_2 are composable if and only if $A(m_1)$ and $A(m_2)$ are composable. If m_1 and m_2 are composable, and m_1cm_2 is a composition, then $A(m_1cm_2)=$ $=A(m_1)cA(m_2)$.*

Proof. It is a simple verification. ∎

(4.2) *If G_1 and G_2 are two composable graphs, G_1cG_2 is an alternance graph if and only if G_1 and G_2 are alternance graphs.*

Proof. If G_1 and G_2 are alternance graphs then the preceding property implies that G_1cG_2 is also an alternance graph. Conversely we notice that G_1 (G_2) is isomorphic to a subgraph G_1' of G_1cG_2. Moreover a subgraph of an alternance graph is clearly an alternance graph too. Therefore if G_1cG_2 is an alternance graph, G_1 and G_2 are also alternance graphs. ∎

Let v be a letter of a double occurence word $m=vA_1vB_1$, where A_1 and B_1 are subwords of m. To *switch* m at v is to construct the double occurence word $m*v=$ $=vA_1v\tilde{B}_1$. If $v_1v_2\ldots v_q$, $q>1$, is a word on V we recursively define $m*v_1v_2\ldots v_q=$ $=(m*v_1v_2\ldots v_{q-1})*v_q$. We let $m*\Lambda=m$.

(4.3) *For every double occurence word m on V and every word p on V, we have $A(m*p)=A(m)*p$.*

Proof. It is a simple verification. ∎

(4.4) *If an alternance graph of order $n\geqq 5$ is prime then there exists a single double occurence word (up to cyclic equivalence) which realizes G.*

Proof. If $n=5$ there exists by (3.1) a word p on V such that $H=G*p$ is a 5-cycle. We let a_1, a_2, a_3, a_4, a_5 be the successive vertices of this cycle, and for each $i=2, 3, 4, 5$ we prove the existence of precisely one double occurence word m_i' (up to cyclic equivalence) such that $A(m_i')$ is equal to the subgraph induced by H on $\{a_1, a_2, ..., a_i\}$. Since $a_1 a_2 \in E(H)$ we have $m_2'=a_1 a_2 a_1 a_2$. The word m_3' must be obtained by inserting in m_2' two occurences of a_3 in such a way that $a_2 a_3$ is an alternance, and $a_1 a_3$ is not an alternance. There are two possibilities $a_1 a_3 a_2 a_3 a_1 a_2$ and $a_1 a_2 a_1 a_3 a_2 a_3$ which are cyclically equivalent; we take the first. Similarly the insertion of two occurences of a_4 in m_3' for obtaining m_4' can be made in two ways, $a_1 a_4 a_3 a_4 a_2 a_3 a_1 a_2$ and $a_1 a_3 a_2 a_4 a_3 a_4 a_1 a_2$ which are cyclically equivalent; we take the first case. Finally there is a single possibility for inserting two occurences of a_5 in m_4' in order to obtain $H=A(m_5')$; this is $m_5'=a_5 a_1 a_4 a_5 a_3 a_4 a_2 a_3 a_1 a_2$. Finally let us consider two double occurence words m and n such that $A(m)=A(n)=G$. We have $A(m*p)=A(m)*p=G*p=H$, and thus $m*p=m_5'$. Similarly $n*p=m_5'$. Therefore $m=n=m_5'*p$, and the property is proved when $n=5$.

Before proving the result for $n>5$ we introduce the notation $V_1(x)$ for the set of the letters occuring precisely once in a word x. We notice that

(i) $V_1(xy)=V_1(x) \triangle V_1(y)$ *for every pair of words* x, y.

For $n>5$ we make an induction. Applying (3.2) there exists a word p on V and a vertex $v \in V$ such that $H=G*p \setminus v$ is prime. By induction there exists a single double occurence word m on $V-v$ such that $A(m)=H$. Therefore if m_1 and m_2 are double occurence words such that $A(m_1)=A(m_2)=G*p$, we can find words A, B, C, D (some of them may be equal to Λ) such that $m=ABCD$, $m_1=vABvCD$, $m_2=AvBCvD$. The alternances containing v in m_1 are the letters of $V_1(AB)=$ $=V_1(CD)$, and the alternances containing v in m_2 are those of $V_1(BC)=V_1(AD)$. Therefore (i) implies $V_1(A)=V_1(C)$ and $V_1(B)=V_1(D)$, and so it appears that m is a composition of the double occurence words $sAsC$ and $sBsD$ with the marker s. Since $A(m)$ is a prime graph this implies either $|V(AC)| \leqq 1$ or $|V(BD)| \leqq 1$. Let us suppose $|V(AC)| \leqq 1$, the case $|V(BD)| \leqq 1$ being similar. If $|V(C)|=0$ and $|V(A)|=1$, then we can write $A=xx$ where x is some letter, $m_1=vxxBvD$, and so it appears that x is an isolated vertex of $A(m_1)$, a contradiction since $G*p$ is connected. If $|V(C)|=|V(A)|=1$ then we can write $A=C=x$ for some letter x, $m_1=vxBvxD$, and so it appears that $\{v, x\}$ is a split of m_1, again a contradiction. Finally it remains the single case $|V(A)|=|V(C)|=0$ which implies $m_1=m_2$. From this equality we prove the existence of a single word m' such that $A(m')=G$. ∎

5. Recognizing and realizing an alternance graph

Theorem 5.1. *If G is a graph of order n there exists a polynomial algorithm for recognizing wherther G is an alternance graph and constructing in the affirmative a double occurence word realizing G. This algorithm recquires $O(n^5)$ time and $O(n^2)$ space.*

Before proving this theorem we need a result which comes from the theory of isotropic systems—(9.4) of [4].

Lemma 5.2. *If G is a graph, v is a vertex of G and w is a neighbour of v, then every elementary i-minor of G at v is locally equivalent to either $G \setminus v$, $G*v \setminus v$ or $G*vwv \setminus v$.* ∎

Corollary 5.3. *For every prime graph G of order $n>5$ there exists a polynomial algorithm for finding a pair (v, p) satisfying:*

(i) $v\in V(G)$,

(ii) *p is a word on V and either $p=\Lambda$ or $p=v$ or $p=vwv$ where w is an arbitrary neighbour of v,*

(iii) *$G*p\setminus v$ is prime.*

Moreover the time-complexity is $O(n^4)$ and the space-complexity is $O(n^2)$.

Proof. We represent G by its adjacency-matrix, which gives space-complexity $O(n^2)$. The algorithm enumerates the vertices of G, and for each of them computes and tests for primality the three elementary i-minors $g*p\setminus v$ for $p=\Lambda, v, vwv$. Computing an elementary i-minor may need three local complementation, which recqires at most $O(n^2)$ time, and testing for primality by Cunningham's algorithm recquires $O(n^3)$ time. This has to be repeated at most $3n$ times, which yields a total time-complexity equal to $O(n^4)$. ∎

Proof of Theorem 5.1. We represent G by its adjacency-matrix, which gives space-complexity $O(n^2)$. Suppose first that G is prime of order $n>5$. The algorithm proceeds in two passes.

The first pass processes a sequence of graphs $g=(g_n, g_{n-1}, \ldots, g_5)$ such that $g_n=G$ and g_{i-1} is an elementary prime i-minor of g_i for $5<i\leq n$. Instead of keeping the sequence g in memory, which would recquire $O(n^3)$ space, the algorithm produces a sequence $v=(v_n, v_{n-1}, \ldots, v_6)$ of vertices of G and a sequence $p=(p_n, p_{n-1}, \ldots$ $\ldots, p_6)$ of words on V such that $g_{i-1}=g_i*p_i\setminus v_i$ for $5<i\leq n$. Where h is a graph, the first pass is initialized with $h=G$ and for each integer i from n down to 6 we determine a pair (v_i, p_i) by using (5.3), we let $G=G*p_i$ and $h=G\setminus\{v_1, v_2, \ldots, v_i\}$. Since the time-complexity of (5.3) is $O(i^4)$, the total time-complexity for the first pass is $O(n)^5$.

At the end of the first pass we obtain a prime graph g_5 of order 5. Following Lemma 3.1 g_5 is locally equivalent to the 5-cycle. Moreover we can translate the first part of the proof of (3.1) into a constant-time algorithm for constructing a double occurence word m_5 realizing g_5.

Starting from m_5 the second pass will construct successively, if possible, a sequence of double occurence words $m=(m_5, m_6, \ldots, m_n)$ realizing respectively the graphs of the sequence $(g_5, g_6, \ldots, g_n)$. Suppose that m_i is known for $5\leq i<n$. Thus following (4.4) m_i is uniquely determined. Since $g_i=g_{i+1}*p_{i+1}\setminus v_{i+1}$, a double occurence word m'_{i+1} realizing $g_{i+1}*p_{i+1}$ must be constructed by inserting two occurences of v_{i+1} in m_i. A "brute force" algorithm can try all possible choices for these insertions, and for each choice deciding whether $n(v_{i+1})=a(v_{i+1})$, where $n(v_{i+1})$ is the neighborhood of v_{i+1} in $g_{i+1}*p_{i+1}$ and $a(v_{i+1})$ is the set of the vertices alternating with v_{i+1} in m'_{i+1}. The length of m_i is equal to $2i$, and thus there are $O(i^2)$ choices for inserting a pair of v_{i+1}'s in m_i. For each choice the equality $n(v_{i+1})=$ $=a(v_{i+1})$ can be tested in linear time. Therefore the research for m'_{i+1} recquires $O(i^3)$ time. A realization of g_{i+1} is $m_{i+1}=m'_{i+1}*\tilde{p}_{i+1}$, and it is clear that m_{i+1} can be constructed from m'_{i+1} in linear time. Therefore the time-complexity for constructing m is $O(n^4)$, and the complete algorithm when G is prime requires $O(n^5)$ time, the time-complexity of the first pass of G by using Cunningham's algorithm. Each complete graph of the decomposition will be realized by a double occurence

word such as $v_1 v_2 \ldots v_p v_1 v_2 \ldots v_p$, and the prime graphs of the decomposition are processed by the two-passes algorithm. The total time-complexity and the total space-complexity remain bounded by the same values. ∎

Conclusion. We have tried to prove the polynomiality of circle graphs in the shortest way. Thus the $O(n^5)$ time-complexity of our algorithm is certainly not optimal. In particular it is certainly possible to refine the exploration of elementary i-minors in following the proof of the reduction theorem. We think that a good refinement would yield an $O(n^3)$ time-complexity. We notice that recognizing circle graphs contains as a particular case the recognition of permutation graphs — see [9] — which can be done in $O(n^3)$ time.

Note. Between the first and second version of this paper, two new polynomial algorithms have been announced for recognizing circle graphs, one by W. Naji [15] and one by C. P. Gabor, W. Hsu, K. J. Supowit [14]. The common feature to these algorithms is to reduce the recognition problem to prime graphs as we do. Naji's paper defines "geometric orientations" of circle graphs and gives necessary and sufficient conditions for a prime graph to have "geometric orientations". These conditions yield a linear system of $O(n^3)$ equations with unknowns in GF(2) which can be solved in $O(n^9)$ time by usual methods. The proof of the main theorem is not given in the announcement. The method by Gabor, Hsu and Supowit assumes the unique realizability of prime circle graphs, but this property is not proven. (It is proven in the present paper). The algorithm is very efficient since its time-complexity is equal to $O(ne)$ where e is the number of edges. This is done by means of an algorithm for decomposing graphs in $O(ne)$ instead of $O(n^3)$ [5] which is not actually described. The main results of the present paper were announced in [13].

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André Bouchet

Université du Maine
72017 Le Mans Cedex
France